An Exposition on Tests for Multivariate Normality

Manuel Rivas, Mrivas03@mit.edu

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1 Introduction

In this course the Shapiro-Wilk test was introduced to test the null hypothesis that a sample $X_1, \dots, X_n$ came from a normally distributed population. Recall the statistic for the Shapiro-Wilk test:

$$W = \frac{\left(\sum_{i=1}^{n} a_i x_{(i)}\right)^2}{\sum_{i=1}^{n} (x_i - \bar{x})^2}$$

where $x_{(i)}$ is the i-th order statistic and the constants a_i are given by

$$(a_1, \dots, a_n) = \frac{m^T V^{-1}}{(m^T V^{-1} V^{-1} m)^{1/2}}$$

where $m = (m_1, \dots, m_n)^T$ and $m_1, \dots, m_n$ are the expected values of the order statistics of i.i.d. random variables sampled from the standard normal distribution, and V is the covariance matrix of those order statistics. We reject H_0 if W is too small. Furthermore, since the Kolmogorov-Smirnov test is consistent against any alternative it can also be used for testing the null hypothesis of normality. The Shapiro-Wilk test is more powerful as it is devised specifically for testing normality.

It is of interest to have available a test for multivariate normality for continuous and discrete multivariate distributions. Note that because of the variety of possible departures from multivariate normality it is desirable to have methods sensitive to different types of departures rather than seeking a single best method. Hence, many tests for multivariate normality have been proposed. In the Wikipedia article for Multivariate Normal Distribution they mention the Cox-Small test and Smith and Jain's adaptation of the Friedman-Rafsky test as tests for the multivariate normal hypothesis. In this paper we survey the Moore and Stubblebine test (a test designed as a result of the Cox-Small test), and the multivariate tests proposed by Quiroz and Manzotti (2001) and Quiroz and Dudley (1991) that claim to have found an omnibus statistic for testing multivariate normality. Also, upon searching for an R implemented test for multivariate normality I came across two libraries available: 1) *mvnorm*, which claims to be an extension of the Shapiro-Wilk test for multivariate data (Note: The documentation for the *mvnorm* package references two papers Domanski (1998) - a Polish paper I was not able to get my hands on, and Royston (1982) which seems to be a companion to the original Shapiro-Wilk test for normality and an algorithm paper describing exact computation of p-values), 2) Energy test of multivariate normality available in the library package *energy*. The documentation for the energy package cites the paper Rizzo and Szekely (2005). The author claims that the mvnorm.test call is a rotation invariant multivariate goodness-of-fit test implemented for testing multivariate normality with estimated parameters. I did, however, come across a message on the R help archive that mentions that the two tests that are available for testing multivariate normality when applied to the same data gave **VERY** different results, but this is not necessarily attributed to be a paradox and can be reasonably explained by the idea that two different tests can have different powers against the same alternative.

2 Background

Quiroz and Manzotti (2001) study the behavior of two statistics for testing multivariate normality which use the notion of *spherical harmonics*, their statistic can be expressed as quadratic forms of the empirical process. The theory behind the statistics was outlined in the paper Quiroz and Dudley (1991). Monte Carlo simulated quantiles of six alternative distributions were used for a comparative power analysis of the two statistics with four other statistics: Mardia's skewness statistic (MS), BHEP tests with parameters $\beta = 1.25$ and $\beta = 15$, and the statistic based on a kernel density estimator of Bowman and Foster (1993).

The first test statistic of Quiroz and Dudley is an improvement of an earlier test of Moore and Stubblebine (1981) that was reported to have good power against departures from normality. The improved statistic takes advantage of the notion of spherical harmonics to detect departure from normality due to non-uniformity of the angular coordinates. The second test also uses the notion of spherical harmonics and uses powers as the function of the radial variable r motivated by the success of statistics based on averaging of the functions of that form to detect different departures from normality.

2.1 Spherical Harmonics

Spherical harmonics are the angular portion of an orthogonal set of solutions to Laplace's equations represented in a system of spherical coordinates. Recall, Laplace's equation

$$\nabla^2 f = 0$$

where ∇^2 is the Laplacian, a scalar differentiable operator defined as the divergence of the gradient such that

$$\Delta = \nabla^2 = \nabla \cdot \nabla = \sum_{i=1}^{q} \frac{\partial^2}{\partial x_i^2}$$

The Wikipedia article on spherical harmonics concentrates on the $q = 3$. Manzotti and Quiroz consider general dimension - the subject of Claus Müller's lecture notes in the theory of regular spherical harmonics in any number of dimension (1966). The usual approach is such that the two or three dimensional problems do not stand out separately. They are regarded as special cases of a more general structure. Müller derives many results solely from the symmetry of the sphere and probe the basic properties of the addition theorem, the representation by a generating function, and the completeness of the entire system.

Let $(x_1, \ldots, x_q)$ be Cartesian coordinates of a Euclidean space of q dimensions. Then we have with

$$|x|^2 = \tau^2 = (x_1)^2 + \cdots + (x_q)^2$$

the representation

$$x = \tau\xi$$

where

$$\xi = (\xi_1, \dots, \xi_q) \quad \text{and} \quad |\xi| = 1$$

represents the system of coordinates of the points on the unit sphere in q dimensions. Let $H_n(x)$ be a homogeneous polynomial of degree n in q dimensions which satisfies $\Delta_q H_n(x) = 0$ then $S_n(\xi) = \frac{1}{\tau^n} H_n(\tau\xi) = H_n(\xi)$ is called a regular spherical harmonic of order n in q dimensions.
The number $N(q, n)$ of linearly independent spherical harmonics of degree n is given by the power series $\frac{1+x}{(1-x)^{q-1}} = \sum_{n=0}^{\infty} N(q, n)x^n$. Letting $S_n\xi = \sum_{j=1}^{N(q,n)} c_j^n S_{n,j}(\xi)$, then there exists $N(q, n)$ linearly independent spherical harmonics $S_{n,j}(\xi)$ of degree n in q dimensions and every spherical harmonic of degree n can be regarded as a linear combination of the $S_{n,j}(\xi)$.

2.1.1 Addition Theorem

Let $S_{n,j}(\xi)$, $j = 1, \dots, N$ be an orthonormal set of spherical harmonics on Ω_q. Then for any two points or vectors ξ and η — $\eta = t\epsilon_q + \sqrt{1-t^2}\eta_{q-1}$ and t is the value of the scalar product of ξ and η and $epsilon_q$ is the resulting vector after applying an orthogonal transformation to the unit vector ξ, i.e. $\epsilon_q = \mathbf{A}\xi$ — on Ω_q the function

$$\phi(\xi \cdot \eta) = \sum_{j=n}^{N(q,n)} S_{n,j}(\xi) S_{n,j}(\eta).$$

This function is a spherical harmonic in ξ or η of degree n. The Addition Theorem states that the function of a spherical harmonic in ξ or η of degree n can also be expressed as

$$\phi(\xi \cdot \eta) = \sum_{j=n}^{N(q,n)} S_{n,j}(\xi) S_{n,j}(\eta) = \frac{N(q, n)}{w_q} P_n(\xi \cdot \eta),$$

where P_n is the Legendre polynomial of degree n and dimension q and w_q is the total surface of the unit sphere given by $\frac{2(\pi)^{\frac{q}{2}}}{\Gamma(\frac{q}{2})}$. The theorem is called the addition theorem since it reduces to the addition theorem for the function $cos\varphi$ in the two-dimensional case after introducing polar coordinates.

2.1.2 Representation Theorem

Since trigonometric functions can be derived by a simple algebraic process from a single one, Müller (1966) gives such a result in the theory of general spherical harmonics in Theorem 3 of his book.
To every degree n, there is a system of N points $\eta_1, \eta_2, \dots, \eta_N$ such that every spherical

harmonic $S_n(\xi)$ can be expressed in the form $S_n(\xi) = \sum_{k=1}^{N(q,n)} a_k P_n(\eta_k \cdot \xi)$. A representation of the Legendre polynomials is made available by Rodrigues' formula

$$P_n(t) = \left(\frac{-1}{2}\right)^n \frac{\Gamma\left(\frac{q-1}{2}\right)}{\Gamma\left(n + \frac{q-1}{2}\right)} \left(1-t^2\right)^{\frac{3-q}{2}} \left(\frac{d}{dt}\right)^n \left(1-t^2\right)^{\frac{n+(q-3)}{2}}$$

and via expansion simplified to

$$P_n(t) = \frac{1}{N(q,n)} \frac{\Gamma(n+q/2)}{\Gamma(q/2)} \frac{2^n}{n!} t^n + \ldots.$$

For further reading about the associated Legendre functions used to get an explicit representation of a system of orthonormal spherical harmonics and for Maxwell's interpretation of the spherical harmonics we recommend Müller (1966).

2.1.3 3-dimensional case

We continue with the three dimensional as a specific example for spherical harmonics since it is one of the dimensions used in the power analysis. In ($q = 3$) dimensions,Laplace's equation in spherical coordinates is

$$\begin{aligned} \nabla^2 f &= \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial f}{\partial r}\right) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial f}{\partial \theta}\right) + \frac{1}{r^2 \sin^2\theta}\frac{\partial^2 f}{\partial \varphi^2} \\ &= 0 \end{aligned}$$

for $f(r,\theta,\phi) = R(r)\Theta(\theta)\Phi(\varphi)$.
The angular portion of Laplace's equation satisfies

$$\frac{\Phi(\varphi)}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta \frac{d\Theta}{d\theta}\right) + \frac{\Theta(\theta)}{\sin^2\theta}\frac{d^2\Phi}{d\varphi^2} + l(l+1)\Theta(\theta)\Phi(\varphi) = 0,$$

where l integers, are eigenvalues of a differential operator w.r.t. Θ. Via separation of variables

$$\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta \frac{d\Theta}{d\theta}\right)\frac{\sin^2\theta}{\Theta(\theta)} + l(l+1)\sin^2\theta = -\frac{1}{\Phi(\varphi)}\frac{d^2\Phi}{\dot{\varphi}^2}.$$

We let $\frac{1}{\Phi(\varphi)}\frac{d^2\Phi}{d\varphi^2} = -m^2$. Then, it follows that

$$\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta \frac{d\Theta}{d\theta}\right)\frac{\sin^2\theta}{\Theta(\theta)} + l(l+1)\sin^2\theta = m^2.$$

and the angular solutions result to be product of trigonometric functions and an associated Legendre function

$$\Upsilon_l^m(\theta,\varphi) = N e^{im\varphi} P_l^m(\cos\theta)$$

where Υ_l^m is the spherical harmonic function of degree l and order m. P_l^m is the associated Legendre function, N the normalizing constant, and θ and φ colatitude and longitude.

2.2 Test of Moore and Stubblebine

Recall that the statistics proposed by Quiroz and Dudley (1991) are improvements of the test of S. Csörgö (1986) and Moore and Stubblebine (1981).

General Setup. Let $X_1, \ldots, X_n$ be an i.i.d. sample from a probability law P on $\mathbb{R}^q$. The statistic considered tests the hypothesis

$$H_0 : P \in \{N(\mu, \Sigma), \mu = (\mu_1, \ldots, \mu_q)^t \in \mathbb{R}^q, \Sigma = (\sigma_{ij}) \quad \text{a real positive definite } q \times q \text{ matrix}\}$$

Under the null hypothesis, μ_0 and Σ_0 are the true parameters of the distribution, where $\bar{X}$ (the sample mean vector) and $S = \frac{1}{n-1}\sum_{j=1}^{n}\left(X_j - \bar{X}\right)\left(X_j - \bar{X}\right)^T$ (the sample covariance matrix) are the MLE for μ_0 and Σ_0.

The vectors $Y_i = T_n(x_i)$ are the scaled residuals where T_n is the Mahalanobis transformation given by

$$T_n(x) = S^{-1/2}\left(x - \bar{X}\right), x \in \mathbb{R}^q.$$

The statistics defined are *affine invariant* and thus invariant under such transformation. Let I_q be the $q \times q$ identity matrix. For k, a positive integer, let $f_1, \ldots, f_k$ be real valued functions in $L^2(N(0, I_q))$. Let V be the $k \times k$ matrix with i, j entry

$$V(i,j) = \mathbb{E}(f_i f_j) - \mathbb{E}(f_i)\mathbb{E}(f_j)$$

where expectations are taken with respect to the $N(0, I_q)$ distribution. Let $f = (f_1, \ldots, f_k)^t$,

$$\begin{aligned}\nu_n(f_j) &= n^{-1/2}\sum_{j=1}^{n}\left(f_j(Y_j) - \mathbb{E}(f_j)\right), \quad \text{and} \\ \nu_n(\boldsymbol{f}) &= \left(\nu_n(f_1), \ldots, \nu_n(f_k)\right)^t.\end{aligned}$$

The general form of the goodness of fit statistic is

$$Z_n^2 = \nu_n(\boldsymbol{f})^t V^{-1} \nu_n(\boldsymbol{f}).$$

The test proposed by Moore and Stubblebine (1981) is closely related to the graphical procedure referred to by Cox and Small that plots the ordered distances $\left(x_i - \bar{X}\right)' S^{-1}\left(x_i - \bar{X}\right)$ against the expected values of chi-square order statistics. The test is particularly effective with distributions that have heavy tails or broad shoulders.

Moore and Stubblebine Test. Let $X_1, \ldots, X_n$ be i.i.d. observed random variables in $\mathbb{R}^q$. For given μ and Σ and $0 < c_0 < c_1 < \cdots < c_M = \infty$, we can define cells

$$E_i(\mu, \Sigma) = \{x \in \mathbb{R}^q : c_{i-1} \leq (x - \mu)^t \Sigma^{-1}(x - \mu) \leq c_i\}.$$

Since the estimates of our parameters are given by the MLE, there are M data-dependent cells $E_{in} = E_i(\hat{\theta}_n), i = 1, \ldots, M$. The cell probability for E_{in} under $f(\cdot|\theta)$ is

$$p_{in}(\theta) = \int_{E_{in}} f(x|\theta)dx.$$

Denote N_{in} to be the number of $X_1, \ldots, X_n$ falling in E_{in} and the estimated cell probability $\hat{p} = p_{in}(\hat{\theta})$, then the Pearson chi-square statistic is given by

$$\chi^2(\hat{\theta}_n) = \sum_{i=1}^{M} \frac{(N_{in} - n\hat{p}_{in})^2}{n\hat{p}_{in}}.$$

It follows that $\hat{p}_{in}$ is the same as the probability under $N(\bar{X}, S)$, that $\left(x - \bar{X}\right)^t S^{-1} \left(x - \bar{X}\right)$ falls between c_{i-1} and c_i, with q degrees of freedom.
In the general setup normality is tested by way of differences $\int f(y) d(Q_n - N(0, I))(y)$, where Q_n is the observed empirical distribution of the Y_i, f runs over a finite set of functions in L^2 of the form $f(y) = g(r)h(U)$ where r and U are the spherical coordinates of y,

$$r = ||y|| \quad \text{and} \quad U = y/r \quad \text{for} \quad r > 0,$$

where $|| \cdot ||$ is the Euclidean norm and g_j, $j = 1, \ldots, M$, is an indicator function in the Moore and Stubblebine test of a spherical ring or shell in the space of scaled residuals.

3 Manzotti and Quiroz Statistic # 1

In Quiroz and Dudley (1991), the authors state the Moore and Stubblebine test does not detect departures from normality due to nonuniformity of the distribution of the angular coordinates $U_1, \ldots, U_n$. The first statistic proposed in Quiroz and Dudley (1991) aims to correct this weakness in the Moore and Stubblebine test. Quiroz and Dudley propose to use $g_j(r)h_j(U)$ where h_j are spherical harmonics since finite-dimensional subspaces of the statistics are rotationally invariant as a function of y and the goal is to have a statistic that is both affine and rotational invariant. Manzotti and Quiroz (2001) use the statistic to test in dimensions $d = 2, 3, 4, 5$ and Quiroz and Dudley treat the case of dimension $= 2$ where $U = (\cos\varphi, \sin\varphi)$ with g_j as indicated by Moore and Stubblebine previously outlined. The omnibus statistic is the sum of two terms

$$S_n^2 = Z_n^2 + W_n^2$$

where Z_n^2 is the statistic proposed by Moore and Stubblebine, and W_n^2 is a measure of lack of angular uniformity of $U_1, \ldots, U_n$. The example they give is clear and helpful in understanding the cases for dimensions greater than 2. Hence, we outline it here: instead of U take the angular variable φ that the scaled residual makes with the positive x-axis, $0 \le \varphi < 2\pi$. The choice for $h_1, \ldots, h_m$ is the orthonormal basis of spherical harmonics up to degree L in $d = 2$, which are $\sqrt{2}\{\cos\varphi, \sin\varphi, \cos 2\varphi, \sin 2\varphi, \ldots, \cos L\varphi, \sin L\varphi$.
Let $Y_{j1}, \ldots, Y_{jp}$ be the scaled residuals that fall in the jth ring C_j as given in Moore and Stubblebine. Let $\varphi_{j1}, \ldots, \varphi_{jp}$ be the corresponding angles. If the corresponding angles come from a uniform distribution on $[0, 2\pi)$, then the expected values of

$$a_{njk} = P_n(1_{c_j}(r)\cos(k\varphi)) = n^{-1} \sum_{i=1}^{p} \cos(\varphi_{ji})$$

and

$$b_{njk} = P_n(1_{c_j}(r)\sin(k\varphi)) = n^{-1}\sum_{i=1}^{p}\sin(\varphi_{ji})$$

are both 0. Therefore, the statistic as a measure of deviation from angular uniformity in the jth cell is

$$W_{nj}^2 = 2M\sum_{k=1}^{L}(a_{njk}^2 + b_{njk}^2),$$

and the overall measure of lack of angular uniformity in the space of scaled residuals is

$$W_n^2 = \sum_{j=1}^{M} W_{nj}^2$$

What has not been specified is what real-valued functions in $L^2(N(0, I_q))$ should be used for the statistic in higher dimensions.
Manzotti and Quiroz cite Müller (1966) who gives recursive formulas for the computation of an orthonormal basis with respect to the uniform probability measure on the unit sphere of spherical harmonics of each degree. Together with the recursive equations in Müller and Lemma 15 of his book, which provides a system of normalized associated Legendre functions, they worked out explicit formulas for the spherical harmonics of degree one to four, in an orthonormal basis.

3.1 Number of linearly independent spherical harmonics of degree j in dimension q

Denote H_j the set of spherical harmonics of degree j in the orthonormal basis and $G_j = \bigcup_{0\le i\le j} H_j$. The number of linearly independent spherical harmonics of degree j in dimension q is given by

$$LI(q, j) = \binom{q+j-1}{j} - \binom{q+j-3}{j-2}$$

The total number of functions in dimension q for $g \in G_4\ H_0$, where 4 is the maximum degree of spherical harmonics considered, is $\binom{q+3}{4} + \binom{q+2}{3} - 1$.

4 Manzotti and Quiroz Statistic # 2

Quiroz and Dudley, again use orthonormal spherical harmonics for the second statistic, as the functions of the angular variable U. As previously mentioned, powers were used as the functions of the radial variable r since statistics of this form were used succesfully to detect different departures from normality,

$$g_i(r) = r^{e_i}, \quad i \le M$$

for some positive integers $e_1, \dots, e_M$. Manzotti and Quiroz use the $g_i(r)$'s proposed in Quiroz and Dudley: $g_1(r) = r$ and $g_2(r) = r^3$.

5 Performance of the Manzotti, Quiroz, Dudley Statistics

The comparative power analysis of the statistic in dimension 2 to 5 uses Monte Carlo quantiles of 90% and 95% with 10,000 samples of size $n = 20, n = 50, n = 100, n = \infty$ comparing to four other statistics and showed that the first statistic performed well against the short tailed, non-spherically symmetric distribution but did not perform as well as the other statistics overall. Whereas the second statistic had the best performance consistently improving with sample size for the alternatives considered. The alternative distribution considered include: 1) The Gaussian mixture-bimodal $(1/2)N(0, I_q) + (1/2)N(\mu, I_q)$ with $\mu = (3, \dots, 3)^t$, 2) Pearson type II distribution with density proportional to $(1 - ||x^2||)\mathbf{1}_{||x^2||<1}$, 3) The uniform distribution on the unit cube $[0,1]^q$, 4) distribution having i.i.d coordinates with the Logistic distribution, 5) distribution having i.i.d. coordinates with Tukey's distribution with parameters $\lambda = 5.2$, 6) distribution having i.i.d. coordinates with the Student t_3 distribution.

6 Székely and Rizzo test for multivariate normality

Like the Quiroz and Dudley tests for multivariate normality, the Székely and Rizzo test for multivariate normality is a class of rotation invariant goodness-of-fit test for multivariate distributions. The Székely and Rizzo test is based on Euclidean distance between sampled elements and the resulting test is affine invariant and consistent against all nonnormal fixed alternatives as proved in Theorem 2 of Székely and Rizzo. Thus, whenever the sampled population is nonnormal and fixed, $lim_{n\to\infty}\mathbb{P}(\hat{\epsilon}_{n,q} \geq c_{\alpha,n,q}) = 1$ where $c_{\alpha,n,q}$ denotes the constant satisfying $\mathbb{P}(\hat{\epsilon}_{n,q} \geq c_{\alpha,n,q}) = \alpha$.
Let $y_1, \dots, y_n$ be the standardized sample, Z, Z' are i.i.d. standard q-variate normal, and $|| \cdot ||$ the Euclidean norm.
The ϵ-test (energy-test), of multivariate normality with test statistic for q-variate normality is given by

$$\epsilon = n\left(\frac{2}{n}\sum_{i=1}^{n}\mathbb{E}||y_i - Z|| - \mathbb{E}||Z - Z'|| - \frac{1}{n^2}\sum_{i=1}^{n}\sum_{j=1}^{n}||y_i - y_j||\right)$$

where the computing formula for the q-variate normality test statistic is

$$\epsilon_{n,q} = n\left(\frac{2}{n}\sum_{i=1}^{n}\mathbb{E}||y_i - Z|| - 2\frac{\Gamma(\frac{q+1}{2})}{\Gamma(\frac{q}{2})} - \frac{1}{n^2}\sum_{i=1}^{n}\sum_{j=1}^{n}||y_i - y_j||\right),$$

and

$$\mathbb{E}||a - Z|| = \frac{\sqrt{2}\Gamma(\frac{q+1}{2})}{\Gamma(\frac{q}{2})} + \sqrt{\frac{2}{\pi}} \sum_{k=0}^{\infty} \frac{(-1)^k}{k!2^k} \frac{||a||^{2k+2}}{(2k+1)(2k+2)} \frac{\Gamma(\frac{q+1}{2})\Gamma(k+\frac{3}{2})}{\Gamma(k+\frac{q}{2}+1)}.$$

The test rejects H_0 for large vales of ϵ. The ϵ test of multivariate normality is implemented by parametric bootstrap with n replicates as indicated in the documentation of the *energy* package.

7 Comparison

Interestingly, the Gaussian mixture-bimodal (GM) with distribution $(1/2)N(0, I_q)+(1/2)N(\mu, I_q)$ with $\mu = (3, \ldots, 3)^t$ considered in Henze and Wagner (1997) and in Bowman and Foster (1993) and corresponding to Table 3 of Rizzo and Szekely $0.5N_d(0, I) + 0.5N_d(3, I)$ at $d = 3, 5$ and under consideration at $d = 3, 5$ in Manzotti and Quiroz with statistic $Z^2_{1,n}$ has 39% M.C. power at 5% level, sample size $n = 50$ whereas Szekely and Rizzo have 58% M.C. power at 5% level, sample size $n = 50$. At degree $d = 5$, Manzotti and Quiroz statistic $Z^2_{1,n}$ has 28% M.C. power at 5% level, $Z^2_{2,n}$ has 23% M.C. power at 5% level, and Szekely and Rizzo statistic $Z^2_{1,n}$ has 20% M.C. power at 5% level. Unfortunately, most of the alternative distributions selected were Gaussian Mixture bimodal. In Quiroz and Manzotti we observe that several distinct alternatives were used to evaluate power of the statistic and that the performance of the statistic $Z^2_{1,n}$ - which did not have good performance in other distributions - had overall good and better performance with the Gaussian Mixture bimodal than $Z^2_{2,n}$ - which had overall good overall performance when compared across all tested alternative distributions.

For direct comparison we selected three alternative distributions used in Quiroz and Manzotti: 1) Gaussian mixture-bimodal (GM) - $(1/2)N(0, I_q) + (1/2)N(3, I_q)$, 2) Uniform distribution on the unit cube $[0, 1]^q$, and 3) distribution having i.i.d. coordinates with the Logistic distribution. We evaluate the power of each statistic: 1) $Z^2_{1,n}$, 2)$Z^2_{2,n}$, 3) energy statistic of Szekely and Rizzo, and 4) Multivariate Shapiro Wilk, on the set of alternatives considered, from sets of 1000 samples with the alternative distribution at each dimension $q = 2, 3, 4, 5$ and each sample size $n = 20, 50, 100$. We use the R implementation of the energy test and multivariate Shapiro-Wilk test.

8 Conclusion

It seems as if the Multivariate Shapiro-Wilk test exhibits good power against alternatives that are Gaussian Mixture. I find my numbers to not agree with the power reported by Szekely and Rizzo in their paper for the Gaussian Mixture distribution. I also find behavior that are kind of suspect with the Shapiro-Wilk method, such as increase power with smaller sample size for $\mathbf{U}[0, 1]^q$ and increase power across all alternatives for increasing q. On the other hand, the multivariate energy test exhibits consistent power against the alternatives, i.e. power increases with sample size. Both tests exhibit nearly 100% power at $\alpha = .05$ for the Logistic distribution alternative. The Quiroz and Manzotti statistic exhibits much better power for small samples with the uniform multivariate alternative distribution. However, with increasing sample size the multivariate energy statistic reaches almost 100% power. The multivariate logistic distribution was generated using the package *evevd* and the normal gaussian distributions were generated with the package *mvtnorm*.

Table 1: Power Analysis for Multivariate Shapiro-Wilk test (GM)

(n,q)	2	3	4	5
20	14	30	58	83
50	12	25	43	69
100	11	21	40	65

Table 2: Power Analysis for Multivariate Energy test (GM)

(n,q)	2	3	4	5
20	10	10	9	8
50	8	9	9	9
100	12	9	9	10

Table 3: Power Analysis for Multivariate Shapiro-Wilk test ($\mathbf{U}[0,1]^q$)

(n,q)	2	3	4	5
20	5	13	29	58
50	4	3	9	20
100	5	3	5	9

Table 4: Power Analysis for Multivariate Energy test ($\mathbf{U}[0,1]^q$)

(n,q)	2	3	4	5
20	19	11	8	6
50	55	45	32	21
100	92	91	83	74

Table 5: Power Analysis for Multivariate Shapiro-Wilk test (Logistic)

(n,q)	2	3	4	5
20	52	65	84	96
50	79	90	95	99
100	97	98	99	100

Table 6: Power Analysis for Multivariate Energy test (Logistic)

(n,q)	2	3	4	5
20	41	38	41	43
50	76	84	87	87
100	97	98	99	100

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